Problem 11

Evaluate $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32)$.

Solution

In order to evaluate this product, write the logarithms so that they all have a common base. Consider one of these factors in general and set it equal to x.

$$\log_n(n+1) = x$$

$$n^x = n+1$$

$$\log_2 n^x = \log_2(n+1)$$

$$x\log_2 n = \log_2(n+1)$$

$$x = \frac{\log_2(n+1)}{\log_2 n}$$

As a result,

$$\log_n(n+1) = \frac{\log_2(n+1)}{\log_2 n},$$

which means

$$\begin{aligned}(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{30} 31)(\log_{31} 32) &= \left(\frac{\cancel{\log_2 3}}{\log_2 2}\right)\left(\frac{\cancel{\log_2 4}}{\cancel{\log_2 3}}\right)\left(\frac{\cancel{\log_2 5}}{\cancel{\log_2 4}}\right)\cdots\left(\frac{\cancel{\log_2 31}}{\cancel{\log_2 30}}\right)\left(\frac{\log_2 32}{\cancel{\log_2 31}}\right) \\ &= \frac{\log_2 32}{\log_2 2} \\ &= \frac{5}{1} \\ &= 5.\end{aligned}$$